ECOLE PREPARATOIRE EN SCIENCES ET TECHNIQUES – TLEMCEN

Département de Technologie

CORRIGE DE L'EXAMEN FINAL

Exercice 01 :

1. Centre d'inertie du demi disque seul :

A cause de la symétrie $x_G = 0$ (le centre d'inertie est situé sur l'axe Oy)

$$y_G = \frac{1}{M}\int_s y dm = y_D(\frac{1}{2}disque)$$

$$y_G = \frac{\sigma}{M}\int_s r\sin\theta r dr d\theta = \frac{\sigma}{M}\int_0^R r^2 dr\int_0^\pi \sin\theta d\theta = \frac{2\sigma}{3M}R^3$$

$$y_G = \frac{4R}{3\pi}$$

2. Centre d'inertie de la plaque rectangulaire est : $y_G = \frac{b}{2} = y_P(plaque)$

3. Centre d'inertie du solide :

$$y_G = \frac{M_P.y_P - M_D.y_D}{M_P - M_D} = \frac{\sigma ab.\frac{b}{2} - \frac{\sigma\pi R^2}{2}.\frac{4R}{3\pi}}{\sigma ab - \frac{\sigma\pi R^2}{2}} = \frac{\frac{ab^2}{2} - \frac{2R^3}{3}}{ab - \frac{\pi R^2}{2}}$$

4. La matrice tenseur d'inertie $\Im_{Oxyz}$ du demi-disque :

Par raison de symétrie les produits d'inertie sont nuls.

$$I_{Ox} = \int(y^2 + z^2)dm = \sigma\int_0^R r^3 dr\int_0^\pi \sin^2\theta d\theta = \sigma\frac{R^4\pi}{4.2} = \frac{MR^2}{4}$$

$$I_{Oy} = \int(x^2 + z^2)dm = \sigma\int_0^R r^3 dr\int_0^\pi \cos^2\theta d\theta = \sigma\frac{R^4\pi}{4.2} = \frac{MR^2}{4}$$

$$I_{Oz} = \int(x^2 + y^2)dm = \sigma\int_0^R r^3 dr\int_0^\pi d\theta = \sigma\frac{R^4\pi}{4} = \frac{MR^2}{2}$$

$$\Im_{Oxyz} = \begin{pmatrix} \frac{MR^2}{4} & 0 & 0 \\ 0 & \frac{MR^2}{4} & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{pmatrix}$$

5. La matrice tenseur d'inertie $\Im_{Oxyz}$ de la plaque seule :

Par raison de symétrie les produits d'inerties sont nuls.

$$I_{Ox} = \int(y^2 + z^2)dm = \sigma\int_0^b y^2 dy\int_{-\frac{a}{2}}^{\frac{a}{2}} dx = \sigma\frac{b^3 a}{3} = \frac{Mb^2}{3}$$

$$I_{Oy} = \int(x^2 + z^2)dm = \sigma\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 dx\int_0^b dy = \sigma\frac{a^3 b}{12} = \frac{Ma^2}{12}$$

$$I_{Oz} = \int(x^2 + y^2)dm = \sigma\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 dx\int_0^b dy + \sigma\int_0^b y^2 dy\int_{-\frac{a}{2}}^{\frac{a}{2}} dx = \sigma\frac{a^3 b}{12} + \sigma\frac{b^3 a}{3} = \frac{M}{3}\left(\frac{a^2}{4} + b^2\right)$$

6. La matrice tenseur d'inertie $\Im_{Oxyz}$ du solide :

$$\Im_{Oxyz}=\begin{pmatrix}\frac{Mb^2}{3}-\frac{MR^2}{4} & 0 & 0\\ 0 & \frac{Ma^2}{12}-\frac{MR^2}{4} & 0\\ 0 & 0 & M\left(\frac{a^2}{12}+\frac{b^2}{3}\right)-\frac{MR^2}{2}\end{pmatrix}=\begin{pmatrix}\frac{\sigma ab^3}{3}-\frac{\sigma\pi R^4}{8} & 0 & 0\\ 0 & \frac{\sigma a^3b}{12}-\frac{\sigma\pi R^4}{8} & 0\\ 0 & 0 & \sigma\left(\frac{a^3b}{12}+\frac{ab^3}{3}\right)-\frac{\sigma\pi R^4}{4}\end{pmatrix}$$

7. Le moment d'inertie par rapport à un axe (Δ) passant par les points O et C :

Les coordonnées du point C : $(\frac{a}{2},b)$ avec $\vec{u}=\frac{\overrightarrow{OC}}{\|\overrightarrow{OC}\|}=\frac{\frac{a}{2}\vec{i}+b\vec{j}}{\left(\frac{a^2}{4}+b^2\right)^{\frac{1}{2}}}$

$$I_\Delta=\vec{u}.\Im_{Oxyz}.\vec{u}=\frac{1}{\frac{a^2}{4}+b^2}\begin{pmatrix}\frac{a}{2}\\ b\\ 0\end{pmatrix}\begin{pmatrix}\frac{\sigma ab^3}{3}-\frac{\sigma\pi R^4}{8} & 0 & 0\\ 0 & \frac{\sigma a^3b}{12}-\frac{\sigma\pi R^4}{8} & 0\\ 0 & 0 & \sigma\left(\frac{a^3b}{12}+\frac{ab^3}{3}\right)-\frac{\sigma\pi R^4}{4}\end{pmatrix}\begin{pmatrix}\frac{a}{2}\\ b\\ 0\end{pmatrix}$$

$$I_\Delta=\vec{u}.\Im_{Oxyz}.\vec{u}=\frac{1}{\frac{a^2}{4}+b^2}\left[\frac{\sigma a^3b^3}{12}-\frac{\sigma\pi R^4a^2}{32}+\frac{\sigma a^3b^3}{12}-\frac{\sigma\pi R^4b^2}{8}\right]$$

Exercice 02 :

$(S_0)\equiv(0)$: O*xyz* repère fixe

$(S_1)\equiv(1)$: $Ax_1y_1z_1$ repère lié à la barre AB

$(S_2)\equiv(2)$: $Ox_2y_2z_2$ repère lié à OA avec ($Ox_2//\overrightarrow{OA}$).

On a $\vec{w}_{1/2}=\begin{vmatrix}0\\0\\\dot{\phi}\end{vmatrix}_{1,2}$; $\vec{w}_{2/0}=\begin{vmatrix}0\\0\\\dot{\theta}\end{vmatrix}_{2,0}$ et $\vec{w}_{1/0}=\vec{w}_{1/2}+\vec{w}_{2/0}=\begin{vmatrix}0\\0\\\dot{\theta}+\dot{\phi}\end{vmatrix}_{2,1}$

1. Les vecteurs vitesse de G_1 et G_2

<u>1. Projection Sur (S_0) :</u>

$$\vec{V}(G_1)/_0=\left.\frac{d\overrightarrow{OG_1}}{dt}\right|_0=\frac{d}{dt}\begin{vmatrix}\frac{l_1}{2}\cos\theta\\ \frac{l_1}{2}\sin\theta\\ 0\end{vmatrix}_0=\begin{vmatrix}-\frac{l_1}{2}\dot{\theta}\sin\theta\\ \frac{l_1}{2}\dot{\theta}\cos\theta\\ 0\end{vmatrix}_0$$

$$\vec{V}(G_2)/_0=\left.\frac{d\overrightarrow{OG_2}}{dt}\right|_0=\frac{d}{dt}\begin{vmatrix}l_1\cos\theta+\frac{l_2}{2}\cos(\theta+\phi)\\ l_1\sin\theta+\frac{l_2}{2}\sin(\theta+\phi)\\ 0\end{vmatrix}_0=\begin{vmatrix}-l_1\dot{\theta}\sin\theta-\frac{l_2}{2}(\dot{\theta}+\dot{\phi})\sin(\theta+\phi)\\ l_1\dot{\theta}\cos\theta+\frac{l_2}{2}(\dot{\theta}+\dot{\phi})\cos(\theta+\phi)\\ 0\end{vmatrix}_0$$

<u>2. Projection Sur (S_1):</u>

$$\overrightarrow{OG_1}=\frac{\overrightarrow{OA}}{2}=\begin{vmatrix}\frac{l_1}{2}\cos\phi\\ -\frac{l_1}{2}\sin\phi\\ 0\end{vmatrix}_1\Rightarrow\vec{V}(G_1)/_0=\left.\frac{d\overrightarrow{OG_1}}{dt}\right|_0=\left.\frac{d\overrightarrow{OG_1}}{dt}\right|_1+\vec{\omega}_{1/2}\wedge\overrightarrow{OG_1}$$

$$\overrightarrow{OG}_1 = \frac{\overrightarrow{OA}}{2} = \left|\begin{array}{l} \frac{l_1}{2}\cos\phi \\ -\frac{l_1}{2}\sin\phi \\ 0 \end{array}\right|_1 \Rightarrow \vec{V}(G_1)/_0 = \left.\frac{d\overrightarrow{OG}_1}{dt}\right|_0 = \left.\frac{d\overrightarrow{OG}_1}{dt}\right|_1 + \vec{\omega}_{1/0} \wedge \overrightarrow{OG}_1$$

$$\vec{V}(G_1)/_0 = \left|\begin{array}{c} -\frac{l_1}{2}\dot{\phi}\sin\phi \\ -\frac{l_1}{2}\dot{\phi}\cos\phi \\ 0 \end{array}\right|_1 + \left|\begin{array}{c} 0 \\ 0 \\ \dot{\phi}+\dot{\theta} \end{array}\right|_1 \wedge \left|\begin{array}{c} \frac{l_1}{2}\cos\phi \\ -\frac{l_1}{2}\sin\phi \\ 0 \end{array}\right|_1 = \left|\begin{array}{c} -\frac{l_1}{2}\dot{\phi}\sin\phi + (\dot{\phi}+\dot{\theta})\frac{l_1}{2}\sin\phi \\ -\frac{l_1}{2}\dot{\phi}\cos\phi + (\dot{\phi}+\dot{\theta})\frac{l_1}{2}\cos\phi \\ 0 \end{array}\right|_1 = \left|\begin{array}{c} \dot{\theta}\frac{l_1}{2}\sin\phi \\ \dot{\theta}\frac{l_1}{2}\cos\phi \\ 0 \end{array}\right|_1$$

$$\overrightarrow{OG}_2 = \overrightarrow{OA} + \overrightarrow{AG}_2 = \left|\begin{array}{c} l_1\cos\phi \\ -l_1\sin\phi \\ 0 \end{array}\right|_1 + \left|\begin{array}{c} \frac{l_2}{2} \\ 0 \\ 0 \end{array}\right|_1 = \left|\begin{array}{c} l_1\cos\phi + \frac{l_2}{2} \\ -l_1\sin\phi \\ 0 \end{array}\right|_1 \Rightarrow \vec{V}(G_2)/_0 = \left.\frac{d\overrightarrow{OG}_2}{dt}\right|_0 = \left.\frac{d\overrightarrow{OG}_2}{dt}\right|_1 + \vec{\omega}_{1/0} \wedge \overrightarrow{OG}_2$$

$$\vec{V}(G_2)/_0 = \left|\begin{array}{c} -l_1\dot{\phi}\sin\phi \\ -l_1\dot{\phi}\cos\phi \\ 0 \end{array}\right|_1 + \left|\begin{array}{c} 0 \\ 0 \\ \dot{\theta}+\dot{\phi} \end{array}\right|_1 \wedge \left|\begin{array}{c} l_1\cos\phi + \frac{l_2}{2} \\ -l_1\sin\phi \\ 0 \end{array}\right|_1 = \left|\begin{array}{c} l_1\dot{\theta}\sin\phi \\ l_1\dot{\theta}\cos\phi + \frac{l_2}{2}(\dot{\theta}+\dot{\phi}) \\ 0 \end{array}\right|_1$$

<u>3. Projection Sur (S_2):</u>

$$\overrightarrow{OG}_1 = \left|\begin{array}{c} \frac{l_1}{2} \\ 0 \\ 0 \end{array}\right|_2 \Rightarrow \vec{V}(G_1)/_0 = \left.\frac{d\overrightarrow{OG}_1}{dt}\right|_0 = \left.\frac{d\overrightarrow{OG}_1}{dt}\right|_2 + \vec{\omega}_{2/0} \wedge \overrightarrow{OG}_1 = \left|\begin{array}{c} 0 \\ 0 \\ \dot{\theta} \end{array}\right|_2 \wedge \left|\begin{array}{c} \frac{l_1}{2} \\ 0 \\ 0 \end{array}\right|_2 = \left|\begin{array}{c} 0 \\ \frac{l_1}{2}\dot{\theta} \\ 0 \end{array}\right|_2$$

$$\overrightarrow{OG}_2 = \overrightarrow{OA} + \overrightarrow{AG}_2 = \left|\begin{array}{c} l_1 \\ 0 \\ 0 \end{array}\right|_2 + \left|\begin{array}{c} \frac{l_2}{2}\cos\phi \\ \frac{l_2}{2}\sin\phi \\ 0 \end{array}\right|_2 = \left|\begin{array}{c} l_1 + \frac{l_2}{2}\cos\phi \\ \frac{l_2}{2}\sin\phi \\ 0 \end{array}\right|_2 \Rightarrow \vec{V}(G_2)/_0 = \left.\frac{d\overrightarrow{OG}_2}{dt}\right|_0 = \left.\frac{d\overrightarrow{OG}_2}{dt}\right|_2 + \vec{\omega}_{2/0} \wedge \overrightarrow{OG}_2$$

$$\vec{V}(G_2)/_0 = \left|\begin{array}{c} -\frac{l_2}{2}\dot{\phi}\sin\phi \\ \frac{l_2}{2}\dot{\phi}\cos\phi \\ 0 \end{array}\right|_2 + \left|\begin{array}{c} -\frac{l_2}{2}\dot{\theta}\sin\phi \\ l_1\dot{\theta} + \frac{l_2}{2}\dot{\theta}\cos\phi \\ 0 \end{array}\right|_2 = \left|\begin{array}{c} -\frac{l_2}{2}(\dot{\theta}+\dot{\phi})\sin\phi \\ l_1\dot{\theta} + \frac{l_2}{2}(\dot{\theta}+\dot{\phi})\cos\phi \\ 0 \end{array}\right|_2$$

2. Les vecteurs accélération de G_1 et G_2

<u>1. Projection Sur (S_1):</u>

$$\vec{\Gamma}(G_1)/_0 = \left.\frac{d\vec{V}(G_1)/_0}{dt}\right|_0 = \left.\frac{d\vec{V}(G_1)/_0}{dt}\right|_1 + \vec{w}_{1/0} \wedge \vec{V}(G_1)/_0$$

$$\vec{\Gamma}(G_1)/_0 = \left|\begin{matrix} \ddot{\theta}\frac{l_1}{2}\sin\phi + \frac{l_1}{2}\dot{\theta}\dot{\phi}\cos\phi \\ \ddot{\theta}\frac{l_1}{2}\cos\phi - \frac{l_1}{2}\dot{\theta}\dot{\phi}\sin\phi \\ 0 \end{matrix}\right| +_1 \left|\begin{matrix} 0 \\ 0 \\ \dot{\theta}+\dot{\phi} \end{matrix}\right|_1 \wedge \left|\begin{matrix} \dot{\theta}\frac{l_1}{2}\sin\phi \\ \dot{\theta}\frac{l_1}{2}\cos\phi \\ 0 \end{matrix}\right|_1 = \left|\begin{matrix} \ddot{\theta}\frac{l_1}{2}\sin\phi - \frac{l_1}{2}\dot{\theta}^2\cos\phi \\ \ddot{\theta}\frac{l_1}{2}\cos\phi + \frac{l_1}{2}\dot{\theta}^2\sin\phi \\ 0 \end{matrix}\right|_1$$

$$\vec{\Gamma}(G_2)/_0 = \left.\frac{d\vec{V}(G_2)/_0}{dt}\right|_0 = \left.\frac{d\vec{V}(G_2)/_0}{dt}\right|_1 + \vec{w}_{1/0} \wedge \vec{V}(G_2)/_0$$

$$\vec{\Gamma}(G_2)/_0 = \left|\begin{matrix} \ddot{\theta}l_1\sin\phi + l_1\dot{\theta}\dot{\phi}\cos\phi \\ \ddot{\theta}l_1\cos\phi - l_1\dot{\theta}\dot{\phi}\sin\phi + \frac{l_2}{2}(\ddot{\theta}+\ddot{\phi}) \\ 0 \end{matrix}\right|_1 + \left|\begin{matrix} 0 \\ 0 \\ \dot{\theta}+\dot{\phi} \end{matrix}\right|_1 \wedge \left|\begin{matrix} \dot{\theta}l_1\sin\phi \\ \dot{\theta}l_1\cos\phi + \frac{l_2}{2}(\dot{\theta}+\dot{\phi}) \\ 0 \end{matrix}\right|_1 = \left|\begin{matrix} \ddot{\theta}l_1\sin\phi - \dot{\theta}^2 l_1\cos\phi - \frac{l_2}{2}(\dot{\theta}+\dot{\phi})^2 \\ \ddot{\theta}l_1\cos\phi + \dot{\theta}^2 l_1\sin\phi + \frac{l_2}{2}(\ddot{\theta}+\ddot{\phi}) \\ 0 \end{matrix}\right|_1$$

<u>2. Projection Sur (S_2):</u>

$$\vec{\Gamma}(G_1)/_0 = \left.\frac{d\vec{V}(G_1)/_0}{dt}\right|_0 = \left.\frac{d\vec{V}(G_1)/_0}{dt}\right|_2 + \vec{\omega}_{2/0} \wedge \vec{V}(G_1)/_0 = \left|\begin{matrix} 0 \\ \frac{l_1}{2}\ddot{\theta} \\ 0 \end{matrix}\right|_2 + \left|\begin{matrix} 0 \\ 0 \\ \dot{\theta} \end{matrix}\right|_2 \wedge \left|\begin{matrix} 0 \\ \frac{l_1}{2}\dot{\theta} \\ 0 \end{matrix}\right|_2 = \left|\begin{matrix} -\frac{l_1}{2}\dot{\theta}^2 \\ \frac{l_1}{2}\ddot{\theta} \\ 0 \end{matrix}\right|_2$$

$$\vec{\Gamma}(G_2)/_0 = \left.\frac{d\vec{V}(G_2)/_0}{dt}\right|_0 = \left.\frac{d\vec{V}(G_2)/_0}{dt}\right|_2 + \vec{\omega}_{2/0} \wedge \vec{V}(G_2)/_0$$

$$\vec{\Gamma}(G_2)/_0 = \left|\begin{matrix} -\frac{l_2}{2}(\ddot{\theta}+\ddot{\phi})\sin\phi - \frac{l_2}{2}\dot{\phi}(\dot{\theta}+\dot{\phi})\cos\phi \\ \ddot{\theta}l_1 + \frac{l_2}{2}(\ddot{\theta}+\ddot{\phi})\cos\phi - \frac{l_2}{2}\dot{\phi}(\dot{\theta}+\dot{\phi})\sin\phi \\ 0 \end{matrix}\right|_2 + \left|\begin{matrix} 0 \\ 0 \\ \dot{\theta} \end{matrix}\right|_2 \wedge \left|\begin{matrix} -\frac{l_2}{2}(\dot{\theta}+\dot{\phi})\sin\phi \\ l_1\dot{\theta} + \frac{l_2}{2}(\dot{\theta}+\dot{\phi})\cos\phi \\ 0 \end{matrix}\right|_2$$

$$\vec{\Gamma}(G_2)/_0 = \left|\begin{matrix} -\frac{l_2}{2}(\ddot{\theta}+\ddot{\phi})\sin\phi - \frac{l_2}{2}\dot{\phi}(\dot{\theta}+\dot{\phi})\cos\phi - l_1\dot{\theta}^2 - \frac{l_1 l_2}{2}\dot{\theta}(\dot{\theta}+\dot{\phi})\cos\phi \\ \ddot{\theta}l_1 + \frac{l_2}{2}(\ddot{\theta}+\ddot{\phi})\cos\phi - \frac{l_2}{2}\dot{\phi}(\dot{\theta}+\dot{\phi})\sin\phi - \frac{l_2}{2}\dot{\theta}(\dot{\theta}+\dot{\phi})\cos\phi \\ 0 \end{matrix}\right|_2$$