

ECOLE PREPARATOIRE EN SCIENCES ET TECHNIQUES – TLEMCCEN

Département de Technologie

CORRIGE DE L'EXAMEN FINAL**Exercice 01 :**

1. Centre d'inertie du demi disque seul :

A cause de la symétrie $x_G = 0$ (le centre d'inertie est situé sur l'axe Oy)

$$y_G = \frac{1}{M} \int y dm = y_D \left(\frac{1}{2} \text{ disque} \right)$$

$$y_G = \frac{\sigma}{M} \int r \sin \theta r dr d\theta = \frac{\sigma}{M} \int_0^R r^2 dr \int_0^\pi \sin \theta d\theta = \frac{2\sigma}{3M} R^3$$

$$y_G = \frac{4R}{3\pi}$$

2. Centre d'inertie de la plaque rectangulaire est : $y_G = \frac{b}{2} = y_P$ (plaque)

3. Centre d'inertie du solide :

$$y_G = \frac{M_P \cdot y_P - M_D \cdot y_D}{M_P - M_D} = \frac{\sigma ab \cdot \frac{b}{2} - \frac{\sigma \pi R^2}{2} \cdot \frac{4R}{3\pi}}{\sigma ab - \frac{\sigma \pi R^2}{2}} = \frac{\frac{ab^2}{2} - \frac{2R^3}{3}}{ab - \frac{\pi R^2}{2}}$$

4. La matrice tenseur d'inertie \mathfrak{J}_{Oxyz} du demi-disque :

Par raison de symétrie les produits d'inertie sont nuls.

$$I_{Ox} = \int (y^2 + z^2) dm = \sigma \int_0^R r^3 dr \int_0^\pi \sin^2 \theta d\theta = \sigma \frac{R^4 \pi}{4 \cdot 2} = \frac{MR^2}{4}$$

$$I_{Oy} = \int (x^2 + z^2) dm = \sigma \int_0^R r^3 dr \int_0^\pi \cos^2 \theta d\theta = \sigma \frac{R^4 \pi}{4 \cdot 2} = \frac{MR^2}{4}$$

$$I_{Oz} = \int (x^2 + y^2) dm = \sigma \int_0^R r^3 dr \int_0^\pi d\theta = \sigma \frac{R^4 \pi}{4} = \frac{MR^2}{2}$$

$$\mathfrak{J}_{Oxyz} = \begin{pmatrix} \frac{MR^2}{4} & 0 & 0 \\ 0 & \frac{MR^2}{4} & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{pmatrix}$$

5. La matrice tenseur d'inertie \mathfrak{J}_{Oxyz} de la plaque seule :

Par raison de symétrie les produits d'inerties sont nuls.

$$I_{Ox} = \int (y^2 + z^2) dm = \sigma \int_0^b y^2 dy \int_{-\frac{a}{2}}^{\frac{a}{2}} dx = \sigma \frac{b^3 a}{3} = \frac{Mb^2}{3}$$

$$I_{Oy} = \int (x^2 + z^2) dm = \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 dx \int_0^b dy = \sigma \frac{a^3 b}{12} = \frac{Ma^2}{12}$$

$$I_{Oz} = \int (x^2 + y^2) dm = \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 dx \int_0^b dy + \sigma \int_0^b y^2 dy \int_{-\frac{a}{2}}^{\frac{a}{2}} dx = \sigma \frac{a^3 b}{12} + \sigma \frac{b^3 a}{3} = \frac{M}{3} \left(\frac{a^2}{4} + b^2 \right)$$

6. La matrice tenseur d'inertie \mathfrak{J}_{Oxyz} du solide :

$$\mathcal{S}_{Oxyz} = \begin{pmatrix} \frac{Mb^2}{3} - \frac{MR^2}{4} & 0 & 0 \\ 0 & \frac{Ma^2}{12} - \frac{MR^2}{4} & 0 \\ 0 & 0 & M\left(\frac{a^2}{12} + \frac{b^2}{3}\right) - \frac{MR^2}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sigma ab^3}{3} - \frac{\sigma \pi R^4}{8} & 0 & 0 \\ 0 & \frac{\sigma a^3 b}{12} - \frac{\sigma \pi R^4}{8} & 0 \\ 0 & 0 & \sigma\left(\frac{a^3 b}{12} + \frac{ab^3}{3}\right) - \frac{\sigma \pi R^4}{4} \end{pmatrix}$$

7. Le moment d'inertie par rapport à un axe (Δ) passant par les points O et C :

Les coordonnées du point C : $\left(\frac{a}{2}, b\right)$ avec $\vec{u} = \frac{\overline{OC}}{\|\overline{OC}\|} = \frac{\frac{a}{2}\vec{i} + b\vec{j}}{\left(\frac{a^2}{4} + b^2\right)^{\frac{1}{2}}}$

$$I_{\Delta} = \vec{u} \cdot \mathcal{S}_{Oxyz} \cdot \vec{u} = \frac{1}{\frac{a^2}{4} + b^2} \begin{pmatrix} \frac{a}{2} \\ b \\ 0 \end{pmatrix} \begin{pmatrix} \frac{\sigma ab^3}{3} - \frac{\sigma \pi R^4}{8} & 0 & 0 \\ 0 & \frac{\sigma a^3 b}{12} - \frac{\sigma \pi R^4}{8} & 0 \\ 0 & 0 & \sigma\left(\frac{a^3 b}{12} + \frac{ab^3}{3}\right) - \frac{\sigma \pi R^4}{4} \end{pmatrix} \begin{pmatrix} \frac{a}{2} \\ b \\ 0 \end{pmatrix}$$

$$I_{\Delta} = \vec{u} \cdot \mathcal{S}_{Oxyz} \cdot \vec{u} = \frac{1}{\frac{a^2}{4} + b^2} \left[\frac{\sigma a^3 b^3}{12} - \frac{\sigma \pi R^4 a^2}{32} + \frac{\sigma a^3 b^3}{12} - \frac{\sigma \pi R^4 b^2}{8} \right]$$

Exercice 02 :

(S_0) \equiv (0) : Oxyz repère fixe

(S_1) \equiv (1) : $Ax_1y_1z_1$ repère lié à la barre AB

(S_2) \equiv (2) : $Ox_2y_2z_2$ repère lié à OA avec ($Ox_2 // \overline{OA}$).

$$\text{On a } \vec{w}_{1/2} = \begin{vmatrix} 0 \\ 0 \\ \dot{\phi}_{1,2} \end{vmatrix} ; \vec{w}_{2/0} = \begin{vmatrix} 0 \\ 0 \\ \dot{\theta}_{2,0} \end{vmatrix} \text{ et } \vec{w}_{1/0} = \vec{w}_{1/2} + \vec{w}_{2/0} = \begin{vmatrix} 0 \\ 0 \\ \dot{\theta} + \dot{\phi}_{2,1} \end{vmatrix}$$

1. Les vecteurs vitesse de G_1 et G_2

1. Projection Sur (S_0):

$$\vec{V}(G_1)_{/0} = \frac{d\overline{OG_1}}{dt} \Big|_0 = \frac{d}{dt} \begin{vmatrix} \frac{l_1}{2} \cos \theta \\ \frac{l_1}{2} \sin \theta \\ 0 \end{vmatrix} = \begin{vmatrix} -\frac{l_1}{2} \dot{\theta} \sin \theta \\ \frac{l_1}{2} \dot{\theta} \cos \theta \\ 0 \end{vmatrix} \Big|_0$$

$$\vec{V}(G_2)_{/0} = \frac{d\overline{OG_2}}{dt} \Big|_0 = \frac{d}{dt} \begin{vmatrix} l_1 \cos \theta + \frac{l_2}{2} \cos(\theta + \phi) \\ l_1 \sin \theta + \frac{l_2}{2} \sin(\theta + \phi) \\ 0 \end{vmatrix} = \begin{vmatrix} -l_1 \dot{\theta} \sin \theta - \frac{l_2}{2} (\dot{\theta} + \dot{\phi}) \sin(\theta + \phi) \\ l_1 \dot{\theta} \cos \theta + \frac{l_2}{2} (\dot{\theta} + \dot{\phi}) \cos(\theta + \phi) \\ 0 \end{vmatrix} \Big|_0$$

2. Projection Sur (S_1):

$$\overline{OG_1} = \frac{\overline{OA}}{2} = \begin{vmatrix} \frac{l_1}{2} \cos \phi \\ -\frac{l_1}{2} \sin \phi \\ 0 \end{vmatrix} \Big|_1 \Rightarrow \vec{V}(G_1)_{/0} = \frac{d\overline{OG_1}}{dt} \Big|_0 = \frac{d\overline{OG_1}}{dt} \Big|_1 + \vec{\omega}_{v_2} \wedge \overline{OG_1}$$

$$\overline{\text{OG}}_1 = \frac{\overline{\text{OA}}}{2} = \begin{vmatrix} \frac{l_1}{2} \cos \phi \\ -\frac{l_1}{2} \sin \phi \\ 0 \end{vmatrix}_1 \Rightarrow \vec{V}(G_1)/_0 = \left. \frac{d\overline{\text{OG}}_1}{dt} \right|_0 = \left. \frac{d\overline{\text{OG}}_1}{dt} \right|_1 + \vec{\omega}_{1/0} \wedge \overline{\text{OG}}_1$$

$$\vec{V}(G_1)/_0 = \begin{vmatrix} -\frac{l_1}{2} \dot{\phi} \sin \phi \\ -\frac{l_1}{2} \dot{\phi} \cos \phi \\ 0 \end{vmatrix}_1 + \begin{vmatrix} 0 \\ 0 \\ \dot{\phi} + \dot{\theta}_1 \end{vmatrix}_1 \wedge \begin{vmatrix} \frac{l_1}{2} \cos \phi \\ -\frac{l_1}{2} \sin \phi \\ 0 \end{vmatrix}_1 = \begin{vmatrix} -\frac{l_1}{2} \dot{\phi} \sin \phi + (\dot{\phi} + \dot{\theta}) \frac{l_1}{2} \sin \phi \\ -\frac{l_1}{2} \dot{\phi} \cos \phi + (\dot{\phi} + \dot{\theta}) \frac{l_1}{2} \cos \phi \\ 0 \end{vmatrix}_1 = \begin{vmatrix} \dot{\theta} \frac{l_1}{2} \sin \phi \\ \dot{\theta} \frac{l_1}{2} \cos \phi \\ 0 \end{vmatrix}_1$$

$$\overline{\text{OG}}_2 = \overline{\text{OA}} + \overline{\text{AG}}_2 = \begin{vmatrix} l_1 \cos \phi \\ -l_1 \sin \phi \\ 0 \end{vmatrix}_1 + \begin{vmatrix} \frac{l_2}{2} \\ 0 \\ 0 \end{vmatrix}_1 = \begin{vmatrix} l_1 \cos \phi + \frac{l_2}{2} \\ -l_1 \sin \phi \\ 0 \end{vmatrix}_1 \Rightarrow \vec{V}(G_2)/_0 = \left. \frac{d\overline{\text{OG}}_2}{dt} \right|_0 = \left. \frac{d\overline{\text{OG}}_2}{dt} \right|_1 + \vec{\omega}_{1/0} \wedge \overline{\text{OG}}_2$$

$$\vec{V}(G_2)/_0 = \begin{vmatrix} -l_1 \dot{\phi} \sin \phi \\ -l_1 \dot{\phi} \cos \phi \\ 0 \end{vmatrix}_1 + \begin{vmatrix} 0 \\ 0 \\ \dot{\theta} + \dot{\phi}_1 \end{vmatrix}_1 \wedge \begin{vmatrix} l_1 \cos \phi + \frac{l_2}{2} \\ -l_1 \sin \phi \\ 0 \end{vmatrix}_1 = \begin{vmatrix} l_1 \dot{\theta} \sin \phi \\ l_1 \dot{\theta} \cos \phi + \frac{l_2}{2} (\dot{\theta} + \dot{\phi}) \\ 0 \end{vmatrix}_1$$

3. Projection Sur (S₂):

$$\overline{\text{OG}}_1 = \begin{vmatrix} \frac{l_1}{2} \\ 0 \\ 0 \end{vmatrix}_2 \Rightarrow \vec{V}(G_1)/_0 = \left. \frac{d\overline{\text{OG}}_1}{dt} \right|_0 = \left. \frac{d\overline{\text{OG}}_1}{dt} \right|_2 + \vec{\omega}_{2/0} \wedge \overline{\text{OG}}_1 = \begin{vmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{vmatrix}_2 \wedge \begin{vmatrix} \frac{l_1}{2} \\ 0 \\ 0 \end{vmatrix}_2 = \begin{vmatrix} 0 \\ \frac{l_1}{2} \dot{\theta} \\ 0 \end{vmatrix}_2$$

$$\overline{\text{OG}}_2 = \overline{\text{OA}} + \overline{\text{AG}}_2 = \begin{vmatrix} l_1 \\ 0 \\ 0 \end{vmatrix}_2 + \begin{vmatrix} \frac{l_2}{2} \cos \phi \\ \frac{l_2}{2} \sin \phi \\ 0 \end{vmatrix}_2 = \begin{vmatrix} l_1 + \frac{l_2}{2} \cos \phi \\ \frac{l_2}{2} \sin \phi \\ 0 \end{vmatrix}_2 \Rightarrow \vec{V}(G_2)/_0 = \left. \frac{d\overline{\text{OG}}_2}{dt} \right|_0 = \left. \frac{d\overline{\text{OG}}_2}{dt} \right|_2 + \vec{\omega}_{2/0} \wedge \overline{\text{OG}}_2$$

$$\vec{V}(G_2)/_0 = \begin{vmatrix} -\frac{l_2}{2} \dot{\phi} \sin \phi \\ \frac{l_2}{2} \dot{\phi} \cos \phi \\ 0 \end{vmatrix}_2 + \begin{vmatrix} -\frac{l_2}{2} \dot{\theta} \sin \phi \\ l_1 \dot{\theta} + \frac{l_2}{2} \dot{\theta} \cos \phi \\ 0 \end{vmatrix}_2 = \begin{vmatrix} -\frac{l_2}{2} (\dot{\theta} + \dot{\phi}) \sin \phi \\ l_1 \dot{\theta} + \frac{l_2}{2} (\dot{\theta} + \dot{\phi}) \cos \phi \\ 0 \end{vmatrix}_2$$

2. Les vecteurs accélération de G₁ et G₂

1. Projection Sur (S₁):

$$\bar{\Gamma}(G_1)/_0 = \left. \frac{d\bar{V}(G_1)/_0}{dt} \right|_0 = \left. \frac{d\bar{V}(G_1)/_0}{dt} \right|_1 + \bar{w}_{y_0} \wedge \bar{V}(G_1)/_0$$

$$\bar{\Gamma}(G_1)/_0 = \begin{vmatrix} \ddot{\theta} \frac{l_1}{2} \sin \phi + \frac{l_1}{2} \dot{\theta} \dot{\phi} \cos \phi \\ \ddot{\theta} \frac{l_1}{2} \cos \phi - \frac{l_1}{2} \dot{\theta} \dot{\phi} \sin \phi \\ 0 \end{vmatrix}_{+1} \wedge \begin{vmatrix} \dot{\theta} \frac{l_1}{2} \sin \phi \\ \dot{\theta} \frac{l_1}{2} \cos \phi \\ \dot{\theta} + \dot{\phi} \end{vmatrix}_1 = \begin{vmatrix} \ddot{\theta} \frac{l_1}{2} \sin \phi - \frac{l_1}{2} \dot{\theta}^2 \cos \phi \\ \ddot{\theta} \frac{l_1}{2} \cos \phi + \frac{l_1}{2} \dot{\theta}^2 \sin \phi \\ 0 \end{vmatrix}_1$$

$$\bar{\Gamma}(G_2)/_0 = \left. \frac{d\bar{V}(G_2)/_0}{dt} \right|_0 = \left. \frac{d\bar{V}(G_2)/_0}{dt} \right|_1 + \bar{w}_{y_0} \wedge \bar{V}(G_2)/_0$$

$$\bar{\Gamma}(G_2)/_0 = \begin{vmatrix} \ddot{\theta} l_1 \sin \phi + l_1 \dot{\theta} \dot{\phi} \cos \phi \\ \ddot{\theta} l_1 \cos \phi - l_1 \dot{\theta} \dot{\phi} \sin \phi + \frac{l_2}{2} (\ddot{\theta} + \ddot{\phi}) \\ 0 \end{vmatrix}_1 \wedge \begin{vmatrix} 0 \\ \dot{\theta} + \dot{\phi} \end{vmatrix}_1 = \begin{vmatrix} \dot{\theta} l_1 \sin \phi \\ \dot{\theta} l_1 \cos \phi + \frac{l_2}{2} (\dot{\theta} + \dot{\phi}) \\ 0 \end{vmatrix}_1 = \begin{vmatrix} \ddot{\theta} l_1 \sin \phi - \dot{\theta}^2 l_1 \cos \phi - \frac{l_2}{2} (\dot{\theta} + \dot{\phi})^2 \\ \ddot{\theta} l_1 \cos \phi + \dot{\theta}^2 l_1 \sin \phi + \frac{l_2}{2} (\ddot{\theta} + \ddot{\phi}) \\ 0 \end{vmatrix}_1$$

2. Projection Sur (S_2):

$$\bar{\Gamma}(G_1)/_0 = \left. \frac{d\bar{V}(G_1)/_0}{dt} \right|_0 = \left. \frac{d\bar{V}(G_1)/_0}{dt} \right|_2 + \bar{\omega}_{z_0} \wedge \bar{V}(G_1)/_0 = \begin{vmatrix} 0 \\ \frac{l_1}{2} \ddot{\theta} \\ 0 \end{vmatrix}_2 + \begin{vmatrix} 0 \\ 0 \\ \dot{\theta} \end{vmatrix}_2 \wedge \begin{vmatrix} 0 \\ \frac{l_1}{2} \dot{\theta} \\ 0 \end{vmatrix}_2 = \begin{vmatrix} -\frac{l_1}{2} \dot{\theta}^2 \\ \frac{l_1}{2} \ddot{\theta} \\ 0 \end{vmatrix}_2$$

$$\bar{\Gamma}(G_2)/_0 = \left. \frac{d\bar{V}(G_2)/_0}{dt} \right|_0 = \left. \frac{d\bar{V}(G_2)/_0}{dt} \right|_2 + \bar{\omega}_{z_0} \wedge \bar{V}(G_2)/_0$$

$$\bar{\Gamma}(G_2)/_0 = \begin{vmatrix} -\frac{l_2}{2} (\ddot{\theta} + \ddot{\phi}) \sin \phi - \frac{l_2}{2} \dot{\phi} (\dot{\theta} + \dot{\phi}) \cos \phi \\ \ddot{\theta} l_1 + \frac{l_2}{2} (\ddot{\theta} + \ddot{\phi}) \cos \phi - \frac{l_2}{2} \dot{\phi} (\dot{\theta} + \dot{\phi}) \sin \phi \\ 0 \end{vmatrix}_2 \wedge \begin{vmatrix} 0 \\ \dot{\theta} \end{vmatrix}_2 = \begin{vmatrix} -\frac{l_2}{2} (\dot{\theta} + \dot{\phi}) \sin \phi \\ l_1 \dot{\theta} + \frac{l_2}{2} (\dot{\theta} + \dot{\phi}) \cos \phi \\ 0 \end{vmatrix}_2$$

$$\bar{\Gamma}(G_2)/_0 = \begin{vmatrix} -\frac{l_2}{2} (\ddot{\theta} + \ddot{\phi}) \sin \phi - \frac{l_2}{2} \dot{\phi} (\dot{\theta} + \dot{\phi}) \cos \phi - l_1 \dot{\theta}^2 - \frac{l_1 l_2}{2} \dot{\theta} (\dot{\theta} + \dot{\phi}) \cos \phi \\ \ddot{\theta} l_1 + \frac{l_2}{2} (\ddot{\theta} + \ddot{\phi}) \cos \phi - \frac{l_2}{2} \dot{\phi} (\dot{\theta} + \dot{\phi}) \sin \phi - \frac{l_2}{2} \dot{\theta} (\dot{\theta} + \dot{\phi}) \cos \phi \\ 0 \end{vmatrix}_2$$